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Fibrant extensions of free G-spaces

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ABSTRACT

We show that any equivariant fibrant extension of a compact free G-space is also free. This result allows us to prove that the orbit space of any equivariant fibrant compact space E is also fibrant, provided that E has only one orbit type.

0. Introduction

One can say that the main reason to study equivariant fibrant extensions and G-fibrant spaces is their role in the equivariant strong shape theory (see [8]). But there are also other motives. Let us recall some of them. If G is a compact metrizable group, then all the orbits of any G-space are equivariant fibrant spaces; this follows from the main result of [7]. The notion of a G-fibrant space can be regarded as a generalization of the concept of a G-*ANR*: every G-*ANR* is G-fibrant and, moreover, the inverse limit of any inverse sequence of G-*ANR*s bonded by G-fibrations is G-fibrant too. The G-fibrant spaces as well as G-*ANR*s have equivariant homotopy extension property with respect to all G-pairs (X, A) of metrizable spaces.

The following fundamental result is due to S.A. Antonyan (see [2,4] and [5]): if E is a G-*ANR*, then the orbit space E/G is an *ANR*, provided that G is a compact group. It is natural to suppose that a similar result may hold for G-fibrant spaces. Though, in general context, this remains an open problem, Theorem 5.4 of the present work gives a positive answer for the case of compact G-spaces with orbits only of one type. This theorem is almost a direct consequence of our main result, that is, Theorem 5.1.

1. Preliminaries

Throughout the paper the letter G will denote a compact Hausdorff group; the unit element is denoted by $\{e\}$. In the most part of the paper we work in the category $\mathcal{M}_G$ of *metrizable* G-spaces and G-maps (equivariant maps).

The foundations of the theory of G-spaces (also known as the theory of topological transformation groups) can be found in [9] and [11]. For the convenience of the reader, below we recall some well-known definitions and facts as well as more special ones.

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Let X be a G-space. For $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *isotropy group* at x. For a subgroup H of G, the symbol X^H denotes the *H-fixed point set*; it can be described as follows: $X^H = \{x \in X \mid H \subseteq G_x\}$. It is well known that X^H is an $N(H)$-invariant subset of X (so that X^H can be regarded as an $N(H)$-space), where $N(H)$ is the normalizer of H in G.

By a *free* G-space X we mean a space in which G acts freely, that is $G_x = \{e\}$ for every $x \in X$. In this case we say that X is G-free. The following simple fact will be used in this paper: if Y is G-free and $f : X \to Y$ is any G-map, then X is also G-free (because $G_x \subseteq G_{f(x)}$ for all $x \in X$).

Let N be a closed normal subgroup of G and let X be a G-space. For every point $x \in X$, the set $N(x) = \{nx \mid n \in N\}$ is called the *N-orbit* of x. The set $X/N = \{N(x) \mid x \in X\}$, which is a quotient space of X, is called the *N-orbit space* of X. The space X/N is a G/N-space with the action $gN \cdot N(x) = N(gx)$. Every G-map $f : X \to Y$ induces a G/N-map $f/N : X/N \to Y/N$ given by $(f/N)(N(x)) = N(f(x))$. Observe that f/N can be also considered as a G-map of G-spaces (see Section 2).

Let X and Y be G-spaces. A homotopy $F : X \times I \to Y$, where $I = [0, 1]$, is called a *G-homotopy*, if $F(gx, t) = gF(x, t)$ for all $g \in G$, $x \in X$ and $t \in I$. Thus F is a G-map, considering $X \times I$ as a G-space with the action $g \cdot (x, t) = (gx, t)$. Note that for every $t \in I$ the map $F_t : X \to Y$, $x \mapsto F(x, t)$ is a G-map.

By *G-fibration* we mean a natural equivariant version of a Hurewicz fibration: a G-map $p : E \to B$ is called a G-fibration if it has the equivariant homotopy lifting property with respect to every G-space X (see [11, p. 53]).

By a *G-A(N)R* (or a *G-A(N)R*-space) we mean a G-equivariant absolute (neighborhood) retract for all metrizable G-spaces (see, for instance, [1,2] and [14] for the equivariant theory of retracts). It is known (see [1, Theorem 14]) that a metrizable G-space Y is a *G-ANR* if and only if it is a *G-ANE*, in other words, it has the following extension property: for any G-map $f : A \to Y$, where A is a closed invariant subset of a metrizable G-space X, there exists a G-map $\bar{f} : U \to Y$ such that $\bar{f}|_A = f$, where U is some invariant neighborhood of A in X.

In the present paper we shall use equivariant versions of such concepts as *SSDR*-map, fibrant space and fibrant extension (in the sense of the work of F. Cathey [10]).

The explicit definition of a *G-SSDR*-map can be found in [8]. Here we point out only the following characterization of such G-maps: a G-map $s : A \hookrightarrow X$ is *G-SSDR-map* iff it is a closed G-embedding such that for every G-fibration $p : E \to B$ of *G-ANR*-spaces and for every commutative diagram of G-maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & E \arrow[d, "p"] \\ X \arrow[r, "F"'] & B \end{tikzcd}$$

there exists a filler $\tilde{F} : X \to E$.

Of course, every *G-SDR*-map $s : A \hookrightarrow X$ (that is, the G-map which embeds A in X as a strong deformation G-retract) is a *G-SSDR*-map. It is important to note that for any closed invariant subset A of a G-space X the inclusion $X \times 0 \cup A \times I \hookrightarrow X \times I$ is a *G-SSDR*-map [7, Corollary 3.2].

Let A be an invariant closed subset of a G-space X. An *infinite strong G-deformation* of X on A is a G-map $D : X \times [0, \infty) \to X$ satisfying:

(a) $D(x, 0) = x$ for all $x \in X$,
(b) $D(a, t) = a$ for all $a \in A$ and $t \in [0, \infty)$,
(c) for each invariant neighborhood U of A in X there is $\lambda \in [0, \infty)$ such that $D(X \times [\lambda, \infty)) \subseteq U$.

A closed G-embedding $s : A \hookrightarrow X$ will be called a *G-ISDR*-map if there is an infinite strong G-deformation of X on $s(A)$. It can be easily seen that every *G-ISDR*-map is a *G-SSDR*-map.

A G-space E is called an *equivariant fibrant* space or a *G-fibrant* space if for every *G-SSDR*-map $s : A \hookrightarrow X$ and every G-map $f : A \to E$, there exists a G-map $F : X \to E$ such that $F \circ s = f$.

For instance, every *G-ANR* is a G-fibrant space. Moreover, the inverse limit of every inverse sequence $\{E_i, q_i^j\}$ of *G-ANR*s E_i and G-fibrations q_i^j is a G-fibrant space. Also the following result, proved in [7], can be mentioned: for any closed subgroup H of a compact metrizable group G the homogeneous space G/H is G-fibrant.

If a *G-SSDR*-map $s : E \hookrightarrow \tilde{E}$ is such that $\tilde{E}$ is a G-fibrant space, then it is said that s is a *G-fibrant extension of E*. Every compact metrizable G-space E admits a G-fibrant extension (see [8]). Moreover, there exists a *G-ISDR*-map $s : E \hookrightarrow \tilde{E}$ into some G-fibrant space $\tilde{E}$. In Section 4 we give the construction of $\tilde{E}$ for any *G-ANR*-resolution of E.

2. Some facts about *G*-fibrations and *G*-fibrant spaces

Throughout the paper we will make use of the following: for a given closed normal subgroup N of a group G, every G/N-space X can also be considered as a G-space; if $*$ is the action of G/N on X, then the action $\cdot$ of G on X is defined by $g \cdot x = gN * x$. Consequently, every G/N-map of G/N-spaces always can be regarded as a G-map. Thus we get a functor $\mathcal{M}_{G/N} \to \mathcal{M}_G$. One can verify that this functor is a right adjoint to the N-orbit functor $-/N : \mathcal{M}_G \to \mathcal{M}_{G/N}$. The simple proof of the next proposition is based on this fact; we give it because the proposition is essentially used afterwards.

Proposition 2.1. *Let N be a closed normal subgroup of a compact group G. Then*:

(a) *If E is a G/N-ANR, then E is a G-ANR.*
(b) *If $p : E \to B$ is a G/N-fibration, then p is also a G-fibration.*
(c) *If $s : A \to X$ is a G-SSDR-map, then the induced map*

$$s/N : A/N \to X/N$$

is a G/N-SSDR-map.
(d) *If E is a G/N-fibrant space, then E is a G-fibrant space.*

Proof. Let $p : E \to B$ be a G/N-map and let $s : A \hookrightarrow X$ be a closed G-embedding. Consider the following commutative diagram of G-maps

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "q_A"] \arrow[ddd, "s"'] & & E \arrow[ddd, "p"] \\
& A/N \arrow[ur, "f/N"'] \arrow[d, "s/N"'] & \\
& X/N \arrow[dr, "F/N"] & \\
X \arrow[ur, "q_X"] \arrow[rr, "F"] & & B
\end{tikzcd}$$

and observe that the existence of a filler $\bar{F} : X/N \to E$ is equivalent to the existence of a filler $\tilde{F} : X \to E$.

(a). To prove that E is a G-*ANR* we need only the upper part of the diagram, so we may assume that B is a one-point set with the trivial action of G. Since E is a G/N-*ANR*-space, for some G/N-invariant neighborhood U of A/N in X/N there exists a G/N-map $\bar{f} : U \to E$ such that $\bar{f}|_{A/N} = f/N$. Then $V = q_X^{-1}(U)$ is a G-invariant neighborhood of A in X and $(\bar{f} \circ q_X)|_V : V \to E$ is a G-extension of f. It means that E is a G-*ANE* and hence it is a G-*ANR*.

(b). Let $X = A \times I$ and $s(a) = (a, 0)$. Obviously, $(A \times I)/N$ can be identified with $(A/N) \times I$. If p is a G/N-fibration, then there is a filler $\bar{F} : (A/N) \times I \to E$. Consequently, the G-map $\tilde{F} : A \times I \to E$, given by $\tilde{F} = \bar{F} \circ q_X$, is also a filler of the diagram. This proves that p is a G-fibration.

(c). Suppose that p is a G/N-fibration of G/N-*ANR*s. In order to prove that s/N is a G/N-*SSDR*-map, we must show that there is a filler $\bar{F} : X/N \to E$ (according to the characterization of G-*SSDR*-maps given in Section 1). By (a) and (b), p is also a G-fibration of G-*ANR*s and, since s is a G-*SSDR*, there exists a filler $\tilde{F} : X \to E$. This implies the existence of the filler $\bar{F}$.

(d) follows immediately from (c) and the definition of G-fibrant space. □

The proof of the next statement is similar to the proof of Proposition 2.1; it can be found, for instance, in [7]. It makes use of the fact that the restriction functor, induced by the inclusion $i_H : H \hookrightarrow G$, is a right adjoint to the functor of twisted product $G \times_H -$ (see [11, Ch. I, Proposition 4.3]).

Proposition 2.2. *Let H be a closed subgroup of a compact group G. Then*:

(a) *If E is a G-ANR and G/H is metrizable, then E is an H-ANR.*
(b) *If $p : E \to B$ is a G-fibration, then p is an H-fibration.*
(c) *If $s : A \hookrightarrow X$ is an H-SSDR-map and G/H is metrizable, then the induced G-map*

$$G \times_H s : G \times_H A \to G \times_H X$$

is a G-SSDR-map.
(d) *If E is a G-fibrant space and G/H is metrizable, then E is H-fibrant.*

The condition "G/H is metrizable" is added in the assertions (a), (c) and (d) in order to get a *metrizable* G-space $G \times_H X$ for any metrizable H-space X (see [3, Proposition 3]) and, therefore, to have a well-defined functor $G \times_H - : \mathcal{M}_H \to \mathcal{M}_G$. Note that this restriction is unnecessary for (b).

Proposition 2.3. *Let $p : E \to B$ be a G-fibration. The induced map $p/G : E/G \to B/G$ is a fibration if the orbit projection $\pi_B : B \to B/G$ is a G-fibration.*

Proof. We must show that for the commutative diagram

$$\begin{tikzcd} x \arrow[dd] & X \arrow[r, "f"] \arrow[dd, "\partial_0"'] & E/G \arrow[dd, "p/G"] \\ & & \\ (x,0) & X \times I \arrow[r, "F"'] & B/G \end{tikzcd}$$

there exists a homotopy $\tilde{F} : X \times I \to E/G$ as a filler. Let us construct the following commutative diagram

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dd] & & \tilde{X} \arrow[ll, "f'"'] \arrow[dl, "\tilde{\partial}_0"'] \arrow[dd, "\pi_{\tilde{X}}"] \\ B \arrow[dd, "\pi_B"'] & & \tilde{X} \times I \arrow[ll, "F'"'] \arrow[dd] & \\ & E/G \arrow[dl] & & X \arrow[ll, "f"'] \arrow[dl, "\partial_0"] \\ B/G & & X \times I \arrow[ll, "F"] & \end{tikzcd}$$

where:

(a) $\tilde{X}$ is the pull-back (fibered product) of $X \xrightarrow{f} E/G \xleftarrow{\pi_E} E$ with the projections f' and $\pi_{\tilde{X}}$,
(b) $\tilde{\partial}_0(\tilde{x}) = (\tilde{x}, 0)$,
(c) the map $\tilde{X} \times I \to X \times I$ is defined by $(\tilde{x}, t) \mapsto (\pi_{\tilde{X}}(\tilde{x}), t)$,
(d) the G-map $F' : \tilde{X} \times I \to B$ is obtained as a covering G-homotopy for the G-homotopy $F \circ (\pi_{\tilde{X}} \times id_I) : \tilde{X} \times I \to B/G$, using that π_B is a G-fibration.

Since p is a G-fibration there exists a G-homotopy $\hat{F} : \tilde{X} \times I \to E$ such that $p \circ \hat{F} = F'$ and $\hat{F} \circ \tilde{\partial}_0 = f'$. Observe that the maps $\pi_{\tilde{X}}$ and $\pi_{\tilde{X}} \times id_I$ can be regarded as G-orbit projections. Therefore $\hat{F}$ induces the map $\tilde{F} = \hat{F}/G : X \times I \to E/G$ which is the required homotopy. □

In the rest of the section we indicate some simple facts which will be used in the paper.

Proposition 2.4. *Suppose that the commutative diagram of G-maps*

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

is a pull-back diagram.

(a) *If p is a G-fibration, then p' is a G-fibration*;
(b) *If E, B, B' are G-ANRs and p is a G-fibration, then E' is a G-ANR.*

We omit the proof of Proposition 2.4 because it is quite similar to the proof of the corresponding non-equivariant version; see, for instance [12, Corollary 6.5]. Note that from this proposition (and [8, Proposition 1.3]) it follows that the mapping cocylinder $\mathrm{coCyl}(f)$ is a *G-ANR* if f is a G-map of *G-ANR*s (see also [13, Proposition 2.4]).

It is well known that the orbit projection $E \to E/G$ is a G-fibration (of course, E/G is considered with the trivial action of G) provided that G is a compact Lie group and E has orbits only of one type (see, e.g., [11, p. 54]). Suppose now that G is an arbitrary compact group and N is its closed normal subgroup such that G/N is a Lie group. By the above result, if E is a free G/N-space, the orbit projection $\pi : E \to E/(G/N)$ is a G/N-fibration. Regarding E as a G-space, we can consider π as a G-orbit projection $E \to E/G$ which is a G-fibration by Proposition 2.1(b). We shall use this simple fact in the proof of Theorem 5.1.

3. Resolutions of free G-spaces

Definition 3.1. Let E be a compact G-space. An inverse sequence of G-*ANR* spaces and G-maps $\{E_i, q_i^j\}$ is called a G-*ANR-resolution of* E if

(1) $E = \varprojlim\{E_i, q_i^j\}$,
(2) the family of natural projections $\{q_i : E \to E_i\}$ satisfies the following condition: for every i and any invariant open neighborhood U of $q_i(E)$ in E_i there exists $j \geqslant i$ such that $q_i^j(E_j) \subseteq U$.

The above definition is a special case of the general definition of a G-*ANR*-resolution given in [2] and [6], which can be applied to arbitrary G-spaces. It can be proved that every G-space admits a G-*ANR*-resolution [6, Theorem 1]. In the present paper we are interested in description of the resolutions of free G-spaces.

Definition 3.2. A *pro-Lie sequence* of subgroups of a given group G is a decreasing sequence

$$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots,$$

of closed normal subgroups of G such that $\bigcap_i N_i = \{e\}$ and G/N_i is a Lie group for every i.

It is well known that for every compact metrizable group G exists a pro-Lie sequence (see, e.g., [15, §46]).
The proof of the following proposition is given in [7] (see also [1, Corollary 9]).

Proposition 3.3. *Let $\{N_i\}_{i\in\mathbb{N}}$ be a pro-Lie sequence of subgroups of a compact group G. If E is a G-space then*

$$E = \varprojlim\{E/N_i, p_i^j\},$$

where $p_i^j : E/N_j \to E/N_i$, $j \geqslant i$, are natural projections.

Proposition 3.4. *Let G be a compact metrizable group and let $\{N_i\}_{i\in\mathbb{N}}$ be any pro-Lie sequence of subgroups of G. If a compact G-space E is free, then E admits a G-ANR-resolution $\{E_i, q_i^j\}$ such that each E_i is a free G/N_i-space.*

Proof. If G is a compact Lie group, then any free (metrizable) G-space X can be regarded as a closed invariant subset of some free G-*ANR*. To see this, consider X as an invariant closed subspace of some G-*AR*-space M. Since G is a compact Lie group, there is a tube about any orbit of the G-space M (see [9, Ch. II, §4 and Theorem 5.4]). In particular, for every $x \in X$ the orbit $G(x)$ is a G-retract of some invariant open neighborhood U_x of $G(x)$ in M and therefore the G-space U_x is free because so is $G(x)$. Obviously, $U = \bigcup_{x\in X} U_x$ is an open invariant neighborhood of X in M and the action of G on U is free. Moreover, U is a G-*ANR* being an open invariant subset of a G-*AR*.

According to Proposition 3.3, we can represent E as

$$E = \varprojlim\{E/N_i, p_i^j\},$$

where $p_i^j : E/N_j \to E/N_i$, $j \geqslant i$, are the natural projections. Since, for every i, G/N_i is a compact Lie group and E/N_i is a free G/N_i-space, we can consider E/N_i as a closed invariant subset of some free G/N_i-*ANR*-space U_i.

For each i we shall find an open invariant neighborhood V_i of E/N_i in U_i by induction as follows. Put $V_1 = U_1$ and suppose that V_i is given. Let V_{i+1} be an open invariant neighborhood of E/N_{i+1} in U_{i+1} for which there is a G-extension $f_i^{i+1} : V_{i+1} \to V_i$ of the composition

$$E/N_{i+1} \to E/N_i \hookrightarrow V_i.$$

This extension exists because V_i is a G-*ANE*.

Now, by obvious induction on i and j, we choose a collection of G-spaces $\{W_i^{(j)}\}_{i,j\in\mathbb{N}}$ which satisfies the following conditions:

(1) For each i the family $\{W_i^{(j)}\}_{j\in\mathbb{N}}$ is a basis of open invariant neighborhoods of E/N_i in V_i such that $W_i^{(j+1)} \subset W_i^{(j)}$ for all j;
(2) $W_{i+1}^{(j)} \subseteq (f_i^{i+1})^{-1}(W_i^{(j)})$ for all i and j.

Finally, we put $E_i = W_i^{(i)}$ for each i, and define $q_i^{i+1} : E_{i+1} \to E_i$ as the restriction of f_i^{i+1} to E_{i+1} (note that $f_i^{i+1}(W_{i+1}^{(i+1)}) \subseteq W_i^{(i+1)} \subseteq W_i^{(i)}$). Of course, $q_i^j : E_j \to E_i$ for $j > i$ is defined by $q_i^j = q_i^{i+1} \circ q_{i+1}^{i+2} \circ \cdots \circ q_{j-1}^j$. It is easily checked that $\{E_i, q_i^j\}$ is the required G-*ANR*-resolution of E. $\square$

4. Cotelescope construction

Recall that so-called cotelescopes are used for the construction of fibrant extensions. The detailed description of an equivariant cotelescope can be found in [8]. However, in this paper, we need cotelescopes for a slightly more general situation than in [8]. Therefore in this section we repeat the main steps of the cotelescope construction.

Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots$ be a decreasing sequence of closed normal subgroups of a group G and let $\mathbf{E} = \{E_i, q_i^{i+1}\}$ be an inverse sequence

$$\begin{tikzcd} E_1 & E_2 \arrow[l, "q_1^2"'] & E_3 \arrow[l, "q_2^3"'] & \cdots \arrow[l] & E_i \arrow[l] & E_{i+1} \arrow[l, "q_i^{i+1}"'] & \cdots \arrow[l] \end{tikzcd}$$

in which E_i is a G/N_i-space and q_i^{i+1} is a G/N_{i+1}-map for each i. Clearly, $\mathbf{E}$ can be regarded also as an inverse sequence of G-spaces and G-maps.

Recall that one can assign to each G/N_i-map q_i^{i+1} a pair of G/N_i-fibrations $p_i : E_{i,i+1} \to E_i$ and $t_{i+1} : E_{i,i+1} \to E_{i+1}$, where $E_{i,i+1} = \operatorname{coCyl}(q_i^{i+1})$. Moreover, t_{i+1} has a right inverse $s_{i+1} : E_{i+1} \hookrightarrow E_{i,i+1}$ which is a G/N_i-*SDR*-map satisfying $q_i^{i+1} = p_i \circ s_{i+1}$. For the given diagram $\mathbf{E}$, we construct a new diagram $T(\mathbf{E})$, changing every arrow q_i^{i+1} by

$$\begin{tikzcd} & E_{i,i+1} \arrow[dl, "p_i"'] \arrow[dr, "t_{i+1}"] & \\ E_i & & E_{i+1} \end{tikzcd}$$

The inverse limit of the diagram $T(\mathbf{E})$ is the *cotelescope* of $\mathbf{E}$; it is denoted by $\operatorname{coTel}(\mathbf{E})$.

For every $n \geqslant 1$, let us denote by $\tilde{E}_n$ the inverse limit of the finite diagram

$$\begin{tikzcd} & E_{12} \arrow[dl, "p_1"'] \arrow[dr, "t_2"] & & E_{23} \arrow[dl, "p_2"'] \arrow[dr, "t_3"] & & \cdots & & E_{n-1,n} \arrow[dl, "p_{n-1}"'] \arrow[dr, "t_n"] & \\ E_1 & & E_2 & & E_3 & \ldots & E_{n-1} & & E_n \end{tikzcd}$$

It can be easily seen that $\tilde{E}_n$ is a G/N_n-space. Moreover, if E_i is a G/N_i-*ANR* for every $i = 1, 2, \ldots, n$, then $\tilde{E}_n$ is a G/N_n-*ANR*. It follows from Propositions 2.4 and 2.1(b) taking into account [8, Proposition 3.1(2)], because $\tilde{E}_n$ (for $n > 2$) can be obtained as a result of applying several times the pull-back construction (first to the pairs (t_i, p_i), $1 < i < n$, and so on).

There is a commutative diagram, which we denote by a symbol $S(\mathbf{E})$,

$$\begin{tikzcd} \tilde{E}_1 & \tilde{E}_2 \arrow[l, "\tilde{q}_1^2"'] & \tilde{E}_3 \arrow[l, "\tilde{q}_2^3"'] & \cdots \arrow[l] & \tilde{E}_n \arrow[l] & \tilde{E}_{n+1} \arrow[l, "\tilde{q}_n^{n+1}"'] & \cdots \arrow[l] \\ E_1 \arrow[u, "\tilde{s}_1 = id"'] & E_2 \arrow[l, "q_1^2"] \arrow[u, "\tilde{s}_2"'] & E_3 \arrow[l, "q_2^3"] \arrow[u, "\tilde{s}_3"'] & \cdots \arrow[l] & E_n \arrow[l] \arrow[u, "\tilde{s}_n"'] & E_{n+1} \arrow[l, "q_n^{n+1}"] \arrow[u, "\tilde{s}_{n+1}"'] & \cdots \arrow[l] \end{tikzcd}$$

satisfying the following conditions (see [8] for details):

(a) $\operatorname{coTel}(\mathbf{E}) = \varprojlim \tilde{\mathbf{E}}$, where $\tilde{\mathbf{E}} = \{\tilde{E}_i, \tilde{q}_i^{i+1}\}$;
(b) $\tilde{q}_i^{i+1} : \tilde{E}_{i+1} \to \tilde{E}_i$ is a G/N_{i+1}-fibration for each i;
(c) $\tilde{s}_i : E_i \hookrightarrow \tilde{E}_i$ is a G/N_i-*SDR*-map for each i.

Clearly, one can change "G/N_i-" for "G-" in (b) (see Proposition 2.1) and (c).

We have already noticed that if each E_i is a G/N_i-*ANR*, then each $\tilde{E}_i$ is also a G/N_i-*ANR*. Hence in this case $\operatorname{coTel}(\mathbf{E})$ is a G-fibrant space because it is the inverse limit of an inverse sequence of G-*ANR*s bonded by G-fibrations.

Now suppose that $\mathbf{E} = \{E_i, q_i^{i+1}\}$ is a G-*ANR*-resolution of a compact G-space E. Then, according to [8, Proposition 3.5], the G/N_i-*SDR*-maps $\tilde{s}_i$ in the diagram $S(\mathbf{E})$ induce a G-map $s : E \hookrightarrow \operatorname{coTel}(\mathbf{E})$ which is a G-*ISDR*-map. Thus we get a G-fibrant extension of E.

5. Main result

Before proving the main theorem of this paper, let us recall the following: if $E \hookrightarrow \tilde{E}$ and $E \hookrightarrow E'$ are two G-fibrant extensions of the same G-space E, then $\tilde{E}$ and E' are G-homotopy equivalent. This fact easily follows from Corollary 3.2 of [7].

Theorem 5.1. *Let G be a compact metrizable group and let E be a free compact G-space. If $s : E \hookrightarrow \tilde{E}$ is any G-fibrant extension of E, then $\tilde{E}$ is a free G-space and the induced map $s/G : E/G \hookrightarrow \tilde{E}/G$ is a fibrant extension of E/G.*

Proof. First we shall show that a G-fibrant extension, satisfying the desired conditions, exists.

Let $\tilde{E} = \mathrm{coTel}(\mathbf{E})$, where $\mathbf{E} = \{E_i, q_i^j\}$ is a G-*ANR*-resolution satisfying Proposition 3.4. In particular, E_i is G/N_i-free for all i. Note that in the diagram $S(\mathbf{E})$ (see Section 4), the space $\tilde{E}_i$ also is G/N_i-free (because, for instance, there is a G/N_i-retraction $\tilde{E}_i \to E_i$) for each i. Recall that $\tilde{E} = \varprojlim\{\tilde{E}_i, \tilde{q}_i^j\}$ and let $\{\tilde{q}_i : \tilde{E} \to \tilde{E}_i\}$ be the family of natural projections. For every $z \in \tilde{E}$ we have $G_z \subseteq G_{\tilde{q}_i(z)} = N_i$ for all i, but $\bigcap_i N_i = \{e\}$ so that $G_z = \{e\}$. Thus $\tilde{E}$ is a free G-space.

Since G is compact, it can be easily seen that $\tilde{E}/G = \varprojlim\{\tilde{E}_i/G, \tilde{q}_i^j/G\}$. By Proposition 2.3, the map $\tilde{q}_i^{i+1}/G$ is a fibration for all i because the G-map $\tilde{q}_i^{i+1}$ and the orbit projection $\pi_{\tilde{E}_i} : \tilde{E}_i \to \tilde{E}_i/G$ are G-fibrations (see the remark at the end of Section 2). Moreover, all the spaces $\tilde{E}_i/G$ are *ANR*s (see [2, Theorem 8], [4, Theorem 6.1] and [5, Theorem 1.1]). Therefore $\tilde{E}/G$ is a fibrant space. Since, for the G-*SSDR*-map $s : E \hookrightarrow \tilde{E}$, the induced map $s/G : E/G \hookrightarrow \tilde{E}/G$ is a *SSDR*-map (by Proposition 2.1(c)), it is a fibrant extension.

Now suppose that $s' : E \hookrightarrow E'$ is an arbitrary G-fibrant extension of E. Then there is a G-homotopy equivalence $\alpha : E' \to \tilde{E}$ and hence E' is a free G-space as well as $\tilde{E}$. Let $\beta : \tilde{E} \to E'$ be some G-homotopy inverse of α and let $D : E' \times I \to E'$ be a G-homotopy with $D_0 = \beta \circ \alpha$ and $D_1 = id_{E'}$. In order to show that E'/G is fibrant we must find, for any given *SSDR*-map $\sigma : A \hookrightarrow X$ and any map $f : A \to E'/G$, a map $\varphi : X \to E'/G$ for which $\varphi \circ \sigma = f$. Since $\tilde{E}/G$ is fibrant, there exists a map $h : X \to \tilde{E}/G$ such that $h \circ \sigma = (\alpha/G) \circ f$. We can construct the following commutative diagram

$$\begin{tikzcd}
 & \tilde{A} \arrow[rr, "\tilde{f}"] \arrow[dl, "\tilde{\sigma}"'] \arrow[dd] & & E' \arrow[dl, "\alpha"'] \arrow[dd, "\pi_{E'}"] \\
\tilde{X} \arrow[rr, "\tilde{h}"] \arrow[dd, "\pi_{\tilde{X}}"'] & & \tilde{E} \arrow[dd] & \\
 & A \arrow[rr, "f"] \arrow[dl, "\sigma"'] & & E'/G \arrow[dl, "\alpha/G"] \\
X \arrow[rr, "h"] & & \tilde{E}/G &
\end{tikzcd}$$

where:

(a) $\tilde{A}$ (with the G-maps $\pi_{\tilde{A}} : \tilde{A} \to A$ and $\tilde{f} : \tilde{A} \to E'$) is the pull-back of $A \xrightarrow{f} E'/G \longleftarrow E'$,

(b) $\tilde{X}$ (with the G-maps $\pi_{\tilde{X}}$ and $\tilde{h}$) is the pull-back of $X \xrightarrow{h} \tilde{E}/G \longleftarrow \tilde{E}$,

(c) $\tilde{\sigma} : \tilde{A} \hookrightarrow \tilde{X}$ is the unique map which exists by the pull-back property.

It is easy to see that $\tilde{\sigma}$ is a closed G-embedding as well as σ. Therefore we can assume, with no loss of generality, that A and $\tilde{A}$ are closed invariant subsets of X and $\tilde{X}$ respectively. Let us define a G-map $F : \tilde{X} \times 0 \cup \tilde{A} \times I \to E'$ as follows: $F(\tilde{x}, 0) = (\beta \circ \tilde{h})(\tilde{x})$ for $(\tilde{x}, 0) \in \tilde{X} \times 0$ and $F(\tilde{a}, t) = D(\tilde{f}(\tilde{a}), t)$ for $(\tilde{a}, t) \in \tilde{A} \times I$. F is well-defined because

$$(\beta \circ \tilde{h})(\tilde{a}) = (\beta \circ \tilde{h})\big(\tilde{\sigma}(\tilde{a})\big) = (\beta \circ \alpha \circ \tilde{f})(\tilde{a}) = D\big(\tilde{f}(\tilde{a}), 0\big)$$

for all $\tilde{a} \in \tilde{A}$. Since the inclusion $\tilde{X} \times 0 \cup \tilde{A} \times I \hookrightarrow \tilde{X} \times I$ is a G-*SSDR*-map (see [7, Corollary 3.2]) and E' is G-fibrant, there is a G-homotopy $\hat{F} : \tilde{X} \times I \to E'$ such that $\hat{F}|_{\tilde{X} \times 0 \cup \tilde{A} \times I} = F$. Let $\tilde{\varphi} = \hat{F}_1$ (that is, $\tilde{\varphi}(\tilde{x}) = \hat{F}(\tilde{x}, 1)$). We state that $\varphi = \tilde{\varphi}/G : X \to E'/G$ is the desired map. Indeed, for every $\tilde{a} \in \tilde{A}$,

$$\tilde{\varphi} \circ \tilde{\sigma}(\tilde{a}) = \hat{F}\big(\tilde{\sigma}(\tilde{a}), 1\big) = F(\tilde{a}, 1) = D\big(\tilde{f}(\tilde{a}), 1\big) = \tilde{f}(\tilde{a}),$$

so that $\tilde{\varphi} \circ \tilde{\sigma} = \tilde{f}$ and therefore $\varphi \circ \sigma = \tilde{\varphi}/G \circ \tilde{\sigma}/G = \tilde{f}/G = f$.

Finally, by Proposition 2.1(c), $s'/G : E/G \to E'/G$ is a *SSDR*-map and hence s'/G is a fibrant extension. $\square$

Lemma 5.2. *If Y is a compact G-fibrant space and N is a closed normal subgroup of G, then the set of N-fixed points Y^N is a G/N-fibrant space (and therefore it is a G-fibrant space).*

Proof. Note that the G-space Y^N can be regarded as a G/N-space because $Y^N/N = Y^N$. Since Y^N is compact there is an infinite strong G/N-deformation of some G/N-fibrant space $Z \supset Y^N$ on Y^N. Clearly, this deformation can be considered in a natural way as an infinite strong G-deformation and hence the embedding $s : Y^N \hookrightarrow Z$ is a G-*SSDR*-map. By the definition of a G-fibrant space there exists a G-map $r : Z \to Y$ such that $r|_{Y^N} = i$, where $i : Y^N \hookrightarrow Y$ is a natural G-embedding. In other words the diagram

$$\begin{tikzcd} Y^N \arrow[r, "i"] \arrow[d, "s"'] & Y \\ Z \arrow[ur, "r"'] & \end{tikzcd}$$

commutes. Now note that for each $z \in Z$ and every $n \in N$ we have $r(z) = r(nz) = nr(z)$ (because all the points of Z, considered as a G-space, are N-fixed). Thus $r(Z) \subseteq Y^N$ and therefore r can be seen as a composition $i \circ r'$, where $r' : Z \to Y^N$ is a G-retraction. Obviously, r' can also be regarded as a G/N-retraction (it is a G/N-map, since $r'(gN * z) = r'(gz) = gr'(z) = gN * r'(z)$). Hence Y^N is G/N-fibrant (because the property to be G-fibrant is clearly preserved under G-retractions). □

Corollary 5.3. *Let H be a closed subgroup of a compact metrizable group G. If X is a G-fibrant space then X^H is an $N(H)/H$-fibrant space.*

Proof. Notice that X is $N(H)$-fibrant by Proposition 2.2(d) and apply Lemma 5.2. □

Theorem 5.4. *Let G be a compact metrizable group. Let E be a compact G-space with only one G-orbit type. If E is G-fibrant, then E/G is a fibrant space.*

Proof. Note first that in the special case when E is a free G-space, E/G is fibrant by Theorem 5.1 because $id : E \to E$ is a G-fibrant extension.

In the general case when all orbits of E have type G/H for some closed subgroup H of G, we can regard E as the twisted product $G \times_{N(H)} E^H$, so that we have the homeomorphism $E/G \approx E^H/N(H)$ (see [9, Ch. II, Theorem 5.9 and Corollary 5.10]). Thus, considering E^H as $N(H)/H$-space, we get that $E/G \approx E^H/(N(H)/H)$. Obviously, E^H is $N(H)/H$-free and moreover, by Corollary 5.3, E^H is an $N(H)/H$-fibrant space. According to the special case indicated at the beginning, $E^H/(N(H)/H)$ is fibrant and hence E/G is a fibrant space. □

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